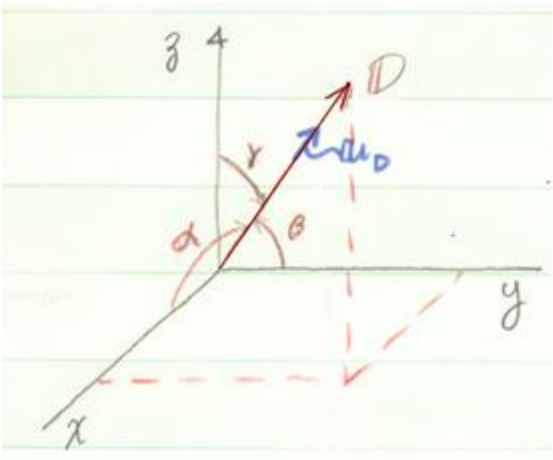
Boas, Chapter 2, section 4 Supplementary Notes:

The scalar component of any vector $\mathbf{R}$ in an arbitrary direction, $\mathbb{D}$, is $\mathbb{R}\cdot\mathbf{u}_D$, where $\mathbf{u}_D$ is a unit vector in the direction D.



We now define directional angles and directional cosines, as shown in the adjacent diagram.

α, β and γ are directional angles for the vector D. That is, they are the angles between the unit vectors **i, j** and **k**, and the vector $\mathbf{D}$.

Define unit vector $\mathbf{u}_D$ as $\mathbf{u}_D = \mathbf{D}/|\mathbf{D}| = \mathbf{D}/D$. Then

$$\mathbf{u}_D = [D_x\mathbf{i} + D_y\mathbf{j} + D_y\mathbf{k}]/D = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k},$$

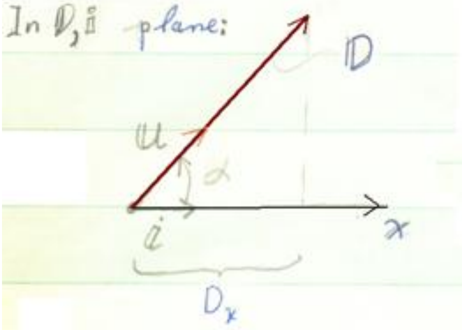
where $u_x = D_x/D$, $u_y = D_y/D$ and $u_z = D_z/D$. These are operational equations, that is, they are how we actually compute the unit vector componets.

Now u_x, u_y and u_z are actually the directional cosines of $\mathbf{D}$.

Proof: $\mathbf{u}_D = [D_x\mathbf{i} + D_y\mathbf{j} + D_z\mathbf{k}]/D = \frac{\mathrm{Dx}}{D}\boldsymbol{i} + \frac{\mathrm{Dy}}{D}\boldsymbol{j} + \frac{\mathrm{Dz}}{D}\boldsymbol{k} = \frac{\mathbf{D}\cdot\mathbf{i}}{D}\boldsymbol{i} + \frac{\mathbf{D}\cdot\mathbf{j}}{D}\boldsymbol{j} + \frac{\mathbf{D}\cdot\mathbf{k}}{D}\boldsymbol{k}$

$$\mathbf{u}_D = \frac{\mathrm{D}\cos\alpha}{D}\boldsymbol{i} + \frac{\mathrm{D}\cos\beta}{D}\boldsymbol{j} + \frac{\mathrm{D}\cos\gamma}{D}\boldsymbol{k}$$

In D, i plane:



$\mathbf{u}_D = \cos\alpha\, \mathbf{i} + \cos\beta\, \mathbf{j} + \cos\gamma\, \mathbf{k}$

or $\mathbf{u}_D = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$

$l = \cos\alpha \qquad m = \cos\beta \qquad n = \cos\gamma$

Example Find l, m, and n for the vector $A = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$A = \sqrt{25+9+4} = \sqrt{38} = 6.164$

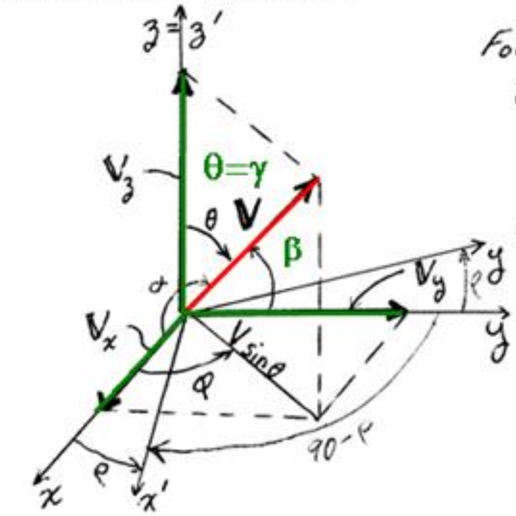
$l = \frac{A_x}{A} = \frac{5}{\sqrt{38}} = 0.811$

$m = A_y/A = 3/\sqrt{38} = 0.487$

$n = A_z/A = \frac{-2}{\sqrt{38}} = -0.324$

Hence, $\mathbf{u}_A = 0.811\mathbf{i} + 0.487\mathbf{j} - 0.324\mathbf{k}$

Vectors in 3 dimensions



$$\mathbf{V} = \mathbf{V}_x + \mathbf{V}_y + \mathbf{V}_z$$

For spherical coordinates, r, θ, ϕ.

$\theta = \gamma$ but $\phi \neq \alpha$ or β.

$$|\mathbf{V}_x| = V_x = \mathbf{V}\cdot\mathbf{i} = V\cos\alpha \quad \mathbf{V}_x = V_x\mathbf{i}$$
$$V_y = \mathbf{V}\cdot\mathbf{j} = V\cos\beta \quad \mathbf{V}_y = V_y\mathbf{j}$$
$$V_z = \mathbf{V}\cdot\mathbf{k} = V\cos\gamma \quad \mathbf{V}_z = V_z\mathbf{k}$$

In terms of spherical coordinate angles:

$$V_x = (V\sin\theta)\cos\phi$$
$$V_y = (V\sin\theta)\sin\phi$$
$$V_z = V\cos\theta$$

Verify the validity of these equations from the geometry in the diagram. To find the component of any vector along any direction, multiply that vector by the cosine of the angle between the vector and the direction. For example, the cosine of the angle between Vsinθ and the y-axis is the cosine of 90- ϕ, which is sine ϕ.

Find $V_{x'}, V_{y'}, V_{z'}$, where the prime coordinate system is rotated about z-axis by angle ρ. Now $V' = RV$ where R is the rotational matrix, which we now find.

$$V_{x'} = V_x\cos\rho + V_y\sin\rho + V_z\cdot 0$$
$$V_{y'} = -V_x\sin\rho + V_y\cos\rho + V_z\cdot 0$$
$$V_{z'} = V_x\cdot 0 + V_y\cdot 0 + V_z\cdot 1$$

The rotation leaves V_z invariant. Hence, in matrix form:

$$V' = \begin{pmatrix} V_{x'} \\ V_{y'} \\ V_{z'} \end{pmatrix} = \begin{pmatrix} \cos\rho & \sin\rho & 0 \\ -\sin\rho & \cos\rho & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

Invoking the concept in the box above for a rotation, we may write:

$$V_{x'_k} = \sum_{i=1}^{n} V_{x_i}\cos\delta_i$$

where n is the number of dimensions and the δ_i are the angles between the vector componets, V_{x_i}, and x'-axis. Actually, the values $V_{x_i}\cos\delta_i$ are the dot products of $\mathbf{V}_{x_i}$ and a unit vector along $V_{x'_k}$.

The angles δ_i are not the angles $\alpha, \beta, \& \gamma$, but are angles between the axes.

E.G. For $k=1$ $x'_k = x$

Hence: $x_1 = x$, $x_2 = y$, $x_3 = z$.

$$V_{x'} = V_x\cos\underbrace{\rho}_{\delta_1} + V_y\cos\underbrace{(90-\rho)}_{\delta_2} + V_z\cos\underbrace{(90^\circ)}_{\delta_3} = V_x\cos\rho + V_y\sin\rho + V_z\cdot 0$$

Solve the following set of linear equations by matrix methods:

$$2x + 3y - z = -3$$
$$x + y + z = 2$$
$$-x + y + 2z = 2$$

First find $|A| = \det A$

$$\det A = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{vmatrix} \overset{\text{replace row 2} \to \text{add row 1 \& 2}}{=} \begin{vmatrix} 2 & 3 & -1 \\ 3 & 4 & 0 \\ -1 & 1 & 2 \end{vmatrix} \overset{\times \text{ row 1 by 2 and add to row 3}}{=} \begin{vmatrix} 2 & 3 & -1 \\ 3 & 4 & 0 \\ 3 & 7 & 0 \end{vmatrix}$$

then $\quad A = -1(3\cdot 7 - 4\cdot 3) = -9$, (expanded about a_{13})

Now find $\hat{A}$ = adjoint of $A = C^T$. First find cofactor matrix C:

$$C = \begin{pmatrix} 1\cdot 2 - (+1\cdot 1) & -[(2\cdot 1) - (+1)(-1)] & 1\cdot 1 - (-1\cdot 1) \\ -[2\cdot 3 - (-1)(+1)] & (2\cdot 2) - (+1)(-1) & -[(2)(1) - (3)(-1)] \\ (3)(1) - (-1)(1) & -[(2)(1) - (-1)(+1) & (2)(1) - (3)(1) \end{pmatrix} = \begin{pmatrix} (2-1) & -(2+1) & (1+1) \\ -[6+1] & (4-1) & -(2+3) \\ (3+1) & -[2+1] & (2-3) \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -3 & 2 \\ -7 & 3 & -5 \\ 4 & -3 & -1 \end{pmatrix} \qquad C^T = \hat{A} = \begin{pmatrix} 1 & -7 & 4 \\ -3 & 3 & -3 \\ 2 & -5 & -1 \end{pmatrix} \qquad A^{-1} = -\frac{1}{9} C^T$$

$$A^{-1} k = r \qquad \underbrace{(-9)}_{|A|} r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -7 & 4 \\ -3 & 3 & -3 \\ 2 & -5 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$$

$$r = -\frac{1}{9} \begin{pmatrix} -3 - 14 + 8 \\ 9 + 6 - 6 \\ -6 - 10 - 2 \end{pmatrix} = -\frac{1}{9} \begin{pmatrix} -9 \\ 9 \\ -18 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So $\quad x = 1 \quad y = -1 \quad \& \quad z = 2$

Chapter 13, Section 5

Solve Laplace's Equation in cylindrical coordinates ρ, φ, z.

$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad (1)$$

Try solution $\quad \Phi = R(\rho)\,Q(\varphi)\,Z(z) \quad (2)$

Substitute into (1):

$$QZ\frac{d^2R}{d\rho^2} + \frac{QZ}{\rho}\frac{dR}{d\rho} + \frac{RZ}{\rho^2}\frac{d^2Q}{d\varphi^2} + RQ\frac{d^2Z}{dz^2} = 0 \quad (3)$$

Divide thru by RQZ:

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\frac{dR}{d\rho} + \frac{1}{Q\rho^2}\frac{d^2Q}{d\varphi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \quad (4)$$

The last term is only a function of z and $\therefore$ must be equal to some constant in order for the sum of all terms to be zero regardless of z: $A + B(z) = 0$ for fixed A but z be any value then $B(z) = k^2$

Hence $$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\frac{dR}{d\rho} + \frac{1}{Q\rho^2}\frac{d^2Q}{d\varphi^2} + k^2 = 0 \quad (5)$$

Now $\cdot/\!/\cdot\ \rho^2$: $$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + \frac{1}{Q}\frac{d^2Q}{d\varphi^2} + k^2\rho^2 = 0 \quad (6)$$

Now the term $\frac{1}{Q}\frac{d^2Q}{d\varphi^2}$ is only a function of φ and must be equal to a constant $-\nu^2$

so: $$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} - \nu^2 + k^2\rho^2 = 0 \quad (7)$$

Now $\cdot/\!/\cdot\ R/\rho^2$:

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0 \quad (8)$$

Now change variables to $x = k\rho$

Then $\frac{d}{d\rho} = \frac{d}{dx}\frac{dx}{d\rho} = k\frac{d}{dx}$ and $\frac{d^2}{d\rho^2} = k^2\frac{d^2}{dx^2}$

Substitute into (8):

$$k^2\frac{d^2R}{dx^2} + \frac{k}{x/k}\frac{dR}{dx} + \left(k^2 - \frac{\nu^2}{x^2/k^2}\right)R = 0 \qquad (11)$$

$$k^2\frac{d^2R}{dx^2} + \frac{k^2}{x}\frac{dR}{dx} + \left(k^2 - \frac{\nu^2 k^2}{x^2}\right)R = 0 \qquad (12)$$

$\div$ by k^2:

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0 \qquad (13)$$

This is Bessel's Equation and the solutions are called Bessel Functions. Bessel's equation arises in physics when there is cylindrical symmetry such as in optics or the propagation of electromagnetic waves or fluid dynamics in pipes.

So the complete solution of Laplace's equation reduces to 3 ordinary differential equations:

$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \qquad (14)$$

$$\frac{d^2Q}{d\varphi^2} + \nu^2 Q = 0 \qquad (15)$$

and equation (8) which transforms to Bessel's Equat.

It can be show that the solution to (14) is

$$Z(z) = e^{\pm kz}$$

and the solution to (15) is:

$$Q(\varphi) = e^{\pm i\nu\varphi} = \cos n\varphi \pm i \sin\varphi$$

This procedure for solving the 2nd order partial differential equation is called "separation of variables."

Assume a solution is a power series

$$R(x) = x^{\nu} \sum_{j=0}^{\infty} a_j x^j \qquad (18)$$

In general, ν is not an integer ; ν is called the order of the Bessel Function.

Let us simplify by assuming there is no φ dependence. Then $\nu = 0$ and Bessel's equation becomes:

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + R = 0 \qquad (19)$$

or

$$xR'' + R' + xR = 0 \qquad (20)$$

So we try a solution $R = \sum_{\lambda} a_\lambda x^\lambda$ where we are using the index λ rather than j as in (18). Now substitute R into (20) and carry out the differentiation to obtain

$$\sum_\lambda \left[x\lambda(\lambda-1)a_\lambda x^{\lambda-2} + a_\lambda \lambda x^{\lambda-1} + x a_\lambda x^\lambda \right] = 0 \qquad (21)$$

$$\sum_\lambda \left[x\lambda^2 a_\lambda x^{\lambda-2} - x\lambda a_\lambda x^{\lambda-2} + a_\lambda \lambda x^{\lambda-1} + x a_\lambda x^\lambda \right] = 0 \qquad (22)$$

Now $x x^{\lambda-2} = x^{\lambda-1}$ and $x x^{\lambda} = x^{\lambda+1}$ so (22) becomes:

$$\sum_\lambda \left[\lambda^2 a_\lambda x^{\lambda-1} + a_\lambda x^{\lambda+1} \right] = 0 \qquad (23)$$

Now we must be able to get a recursion relation for the coefficients, otherwise the trial solution is wrong! Expand the summation starting $\lambda=0$

$$a_0 \cdot 0^2 \cdot x^{-1} + a_0 x + a_1 + a_1 x^2 + a_2 2^2 x + a_3 x^3 + a_3 3^2 x^2$$
$$+ a_3 x^4 + \cdots = 0$$

or $\quad a_1 + (a_0 + a_2 2^2)x + (a_1 + a_3 3^2)x^2 + \cdots = 0$

Hence $\quad a_1 = 0 \qquad (a_0 + 4a_2) = 0 \qquad a_1 + 9a_3 = 0 \quad$ etc

That is $$a_\lambda + a_{\lambda+2}(\lambda+2)^2 = 0 \tag{25}$$

or $$a_{\lambda+2} = -a_\lambda/(\lambda+2)^2 \tag{26}$$

This is the recursion relation. Furthermore, when $\lambda = 1$

$$a_3 = -a_1/(3^2) = -0/9$$
$$a_3 = 0$$

Hence, since $a_1 = 0$, the recursion relation propagates the value of a_1 to all the odd coefficients.

We then have

$$R = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots\right] \tag{27}$$

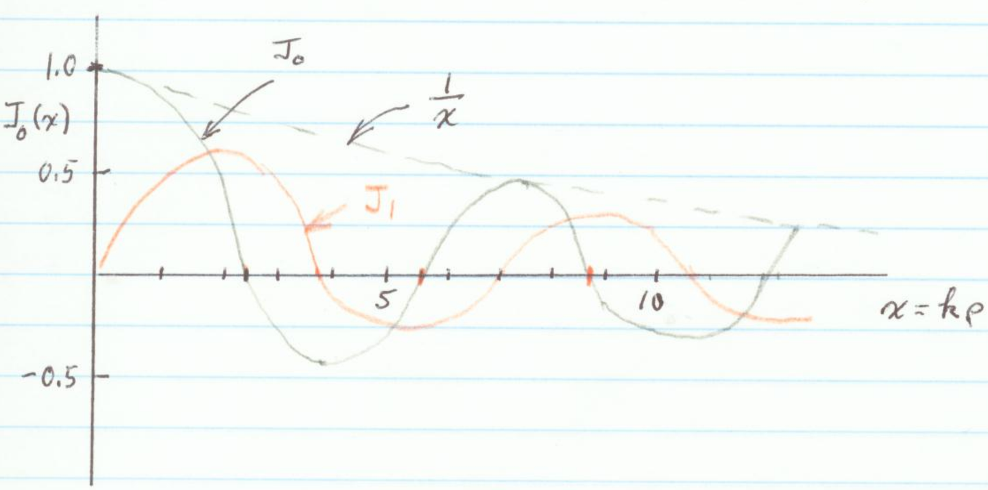
The quantity in the square bracket is called $J_0(x)$, a Bessel function for $\nu = 0$ in (18). This is a particular solution to Bessel's equation.

Another solution, which we shall not derive, is $Y_0(x)$. A complete solution is

$$R(x) = A J_0(x) + B Y_0(x) \tag{28}$$

where $$Y_0(x) = J_0(x)\ln x + \frac{x^2}{4} - \frac{3x^4}{128} + \cdots \tag{29}$$

$Y_0(x)$ blows up at $x=0$ and therefore $B=0$ for problems where $x=0$ in problem at hand.



For $x \gg 1$ $\qquad J_0(x) \simeq \frac{1}{x} \cos x$ (30)

$J_1(x) \simeq \frac{1}{x} \sin x$ (31)

J_0 has roots at $x = 2.40,\ 5.52,\ 8.65,\ 11.79,\ 14.93 \ldots$
and for J_1 $\qquad x = 0.0,\ 3.83,\ 7.02,\ 10.17,\ 13.32 \ldots$

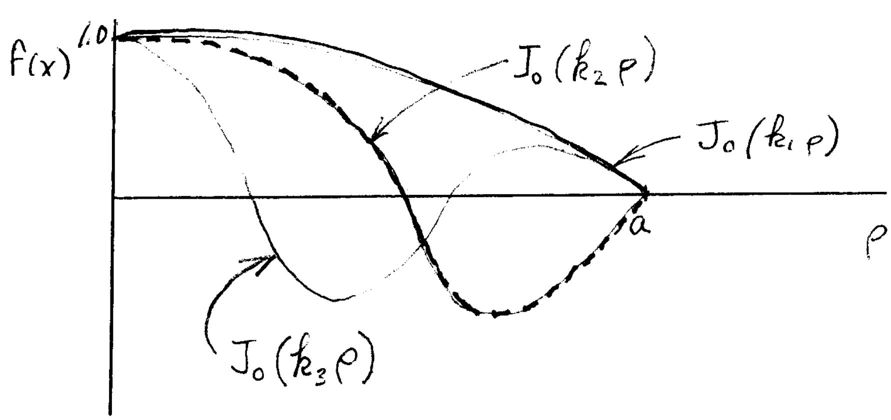
Like the Fourier functions, the Bessel functions may be used to approximate a function over an interval $\rho = 0$ to $\rho = a$, where $x = k\rho$

At $\rho = a$, let $J_0(x) = J_0(k\rho) = J_0(ka) = 0$

That is, the function is 1 at $x = 0$ and 0 at $x = ka$
Different values of k will correspond to different solutions of $\nabla^2 \Phi = 0$, for $z = 0$.
Each value of k corresponds to a different root of J_0 and if J_0 is periodic over the interval $\rho = 0$ to $\rho = a$, then

$k_1 a = 2.40 \qquad k_1 = 2.40/a$
$k_2 a = 5.52 \qquad k_2 = 5.52/a$
$k_3 a = 8.65 \qquad k_3 = 8.65/a$
etc.

That is $k_i = \alpha_n / a$, where α_n is the ith root of $J_0(x)$. Then we have:



Hence, these J_0 functions can be used to approximate a function like $V(x)$ over the interval $0 \to a$.

$$V(x) = \sum_{n=0}^{\infty} A_n J_0(k_i \rho) = \sum_{n=0}^{\infty} A_n J_0\left(\frac{\alpha_n}{a}\rho\right) \quad (32)$$

where $k_i = \alpha_n / a$

Orthogonalization: To make the set of $J_0(x)$ orthogonal, a weighting factor ρ is needed. Then

$$\int_0^a \rho J_0(x) V(x)\, d\rho = \int_0^a \sum A_n \rho J_0(x_i) J_0(x_j)\, d\rho \quad (33)$$

where $x_i = k_i \rho$ and $x_j = k_j \rho$

On the right side of (33):

$$A_n \left\{ \int_a^a \rho J_0^2\left(\frac{\alpha_n}{a}\rho\right) d\rho \right\} = \begin{cases} 0 & i \neq j \\ \frac{a^2}{2} A_n J_1^2(\alpha_n) & i = j \end{cases} \quad (34)$$

Then equating left and right sides of (33):

$$A_n = \frac{\int_0^a V(x) \rho J_0\left(\frac{\alpha_n}{a}\rho\right) d\rho}{\frac{a^2}{2} J_1^2(\alpha_n)} \quad (35)$$

The following relations are needed to find the values of A_n:

$$\int J_1(x)\, dx = -J_0(x) \quad (36)$$

$$\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x) \quad (37)$$

$\int x J_0(x) dx = x J_1(x)$ Values of $\int J_0(x) dx$ are (38)
found tabulated

$$\int x^2 J_0^2\, dx = \frac{1}{2} x^2 \left(J_0^2 + J_1^2\right) \quad (39)$$

$$\int x\, J_1^2(x)\, dx = \frac{1}{2}\left(J_0^2 + J_1^2\right) - x J_0 J_1 \qquad (40)$$

$$I_{n0} = \int x^n J_0(x)\, dx = x^n J_1(x) + (n-1)\, x^{n-1} J_0(x) - (n-1)^2 I_{n-2,0}$$

Expand $1-x^2$ in terms of $J_0(k_n x)$ over $0<x<a$

$$1-x^2 = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_n}{a} x\right)$$

where the α_n's are the zeros of $J_0(x)$.

$$A_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a (1-x^2)\, x\, J_0(k_n x)\, dx = \frac{2}{a^2 J_1^2(\alpha_n)} \left[\int_0^a x J_0(k_n x) dx - \int_0^a x^3 J_0(k_n x) dx \right]$$

change variables: let $k_n x = u$ $\quad du = k_n dx$

then

$$A_n = \frac{2}{a^2 J_1^2(\alpha_n)} \left[\frac{1}{k_n^2} \int u\, J_0(u)\, du - \frac{1}{k_n^4} \int u^3 J_0(u)\, du \right]$$

Use relation below to get:

$$A_n = \frac{2}{a^2 J_1^2} \left[\frac{1}{k_n^2} u J_1(u) - \frac{1}{k_n^4} \left[u(u^2-4) J_1(u) - 2u^2 J_0(u) \right] \right]$$

$$A_n = \frac{2}{a^2 J_1^2} \left[\frac{1}{k_n^2} k_n x J_1(k_n x) - \frac{1}{k_n^4} \left[k_n x (k_n^2 x^2 - 4) J_1(k_n x) - 2 k_n^2 x^2 J_0(k_n x) \right] \right]_0^a$$

$k_n = \alpha_n / a$

$$A_n = \frac{2}{a^2 J_1^2} \left[\frac{a^2}{\alpha_n^2} \left(\frac{\alpha_n}{a}\right) x\, J_1\left(\frac{\alpha_n}{a} x\right) - \frac{a^4}{\alpha_n^4} \left[\frac{\alpha_n}{a} x \left(\frac{\alpha_n^2}{a^2} x^2 - 4 \right) \right] J_1\left(\frac{\alpha_n}{a} x\right) - 2 \left(\frac{\alpha_n}{a}\right)^2 x^2 J_0\left(\frac{\alpha_n}{a} x\right) \right]_0^a$$

Now $J_0(\alpha_n) = 0$, so at $x=a$ minus $x=0$ we have:

$$A_n = \frac{2}{a^2 J_1^2} \left[\frac{a^2}{\alpha_n} J_1(\alpha_n) - \left[\frac{a^4}{\alpha_n} - \frac{4a^4}{\alpha_n^3} \right] J_1(\alpha_n) - 0 - 0 \right]$$

$$A_n = \frac{2}{J_1} \left[\frac{1}{\alpha_n} - \frac{1}{\alpha_n} \left(a^2 - \frac{4a^2}{\alpha_n^2} \right) \right] = \frac{2}{\alpha_n J_1} \left[1 - a^2 \left(1 - \frac{4}{\alpha_n^2} \right) \right]$$

$$A_n = \frac{2}{\alpha_n^3 J_1} \left[\alpha_n^2 (1-a^2) + 4 \right]$$

RJP 1306

For $a=1$, write out the 1st 3 terms of $1-x^2 = \sum A_n J_0$

$J_1(\alpha_1) = 0.52 \qquad J_1(\alpha_2) = -0.34 \qquad J_1(\alpha_3) = 0.27$

$$\int x^n J_0(x)\, dx = x^n J_1(x) + (n-1) x^{n-1} J_0(x) - (n-1)^2 \int x^{n-2} J_0(x)\, dx$$

Moment of Inertia Tensor:

Angular momentum $\qquad \mathbb{L} = I\boldsymbol{\omega} \qquad (1)$

I is the moment of inertia and, in general, is a second rank tensor. In some cases, I is just a scalar.

Consider N particles making up a rigid body. Then

$$\mathbb{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i = \sum_i^N m_i (\mathbf{r}_i \times \mathbf{v}_i) \qquad (2)$$

If the particles make up a rigid body rotating around a fixed axis with angular velocity $\boldsymbol{\omega}$, then

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i \qquad (3)$$

Hence (2) becomes

$$\mathbb{L} = \sum_{i=1}^{N} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \qquad (4)$$

Now $\quad \mathbb{A} \times (\mathbb{B} \times \mathbb{C}) = \mathbb{B}(\mathbb{A}\cdot\mathbb{C}) - \mathbb{C}(\mathbb{A}\cdot\mathbb{B}) \qquad (5)$

Then (4) becomes:

$$\mathbb{L} = \sum_i^N m_i \left[\boldsymbol{\omega}(\mathbf{r}_i \cdot \mathbf{r}_i) - \mathbf{r}_i(\boldsymbol{\omega}\cdot\mathbf{r}_i)\right] \qquad (6)$$

$$= \sum_i^N m_i \left[r_i^2 \boldsymbol{\omega} - \mathbf{r}_i(\boldsymbol{\omega}\cdot\mathbf{r}_i)\right] \qquad (7)$$

Now consider L_x:

$$L_x = \sum_i^N m_i \left[r_i^2 \omega_x - x_i(\omega_x x_i + \omega_y y_i + \omega_z z_i)\right] \qquad (8)$$

Now $\quad r_i^2 \omega_x = (x_i^2 + y_j^2 + z_k^2)\omega_x = (y_i^2 + z_i^2)\omega_x + x_i^2 \omega_x \qquad (9)$

cancel

and $\quad L_x = \sum_i^N m_i \left[(y_i^2 + z_i^2)\omega_x - x_i y_i \omega_y - x_i z_i \omega_z\right] \qquad (10)$

Let $I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$ $\qquad I_{xy} = -\sum_i m_i x_i y_i$ (11, 12)

and $I_{xz} = -\sum_i m_i x_i z_i$ (13)

Then $$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \qquad (14)$$

Now consider L_y:

$$L_y = \sum_i m_i \left[r_i^2 \omega_y - y_i (\omega_x x_i + \omega_y y_i + \omega_z z_i) \right] \qquad (15)$$

Again $r_i^2 \omega_y = x_i^2 \omega_y + y_i^2 \omega_y + z_i^2 \omega_y = (x_i^2 + z_i^2)\omega_y + y_i^2 \omega_y$ (16)

So $$L_y = \sum_i m_i \left[(x_i^2 + z_i^2)\omega_y - x_i y_i \omega_x - \omega_z y_i z_i \right] \qquad (17)$$

Let $I_{yy} = \sum_i m_i (x_i^2 + z_i^2)$, $I_{yx} = -\sum_i m_i x_i y_i$, & $I_{yz} = -\sum_i m_i y_i z_i$ (18)

Then $L_y = I_{yy}\omega_y + I_{yx}\omega_x + I_{yz}\omega_z$ (19)

similarly for L_z

In general $I_{kj} = \sum_i m_i (r_i^2 \delta_{kj} - k_i j_i)$ $\qquad k, j = x, y$ or z (20)

Kronecke delta: $\delta_{kj} \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$

For a continuous body

$$I_{kj} = \int \rho(\mathbf{r}) \left(r^2 \delta_{kj} - x_k x_j \right) d^3x \qquad (21)$$

$k, j = 1, 2, 3$.
But now $x_1 = x$, $x_2 = y$, & $x_3 = z$. Example $k=1$ & $j=1$. Then

$$I_{11} = I_{xx} = \int \rho(\mathbf{r}) \left(r^2 \delta_{11} - x_1 x_1 \right) d^3x \qquad (22)$$

So $I_{xx} = \int \rho(\mathbf{r}) \left(r^2 \delta_{11} - xx \right) d^3x$ (23)

$I_{xx} = \int \rho(\mathbf{r}) \left(x^2 + y^2 + z^2 - x^2 \right) d^3x$ (24)

$I_{xx} = \int \rho(\mathbf{r}) \left(y^2 + z^2 \right) d^3x$ (25)

Compare with (11) $I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$

Now (14), (19) & L_z may be written as set of rows found from

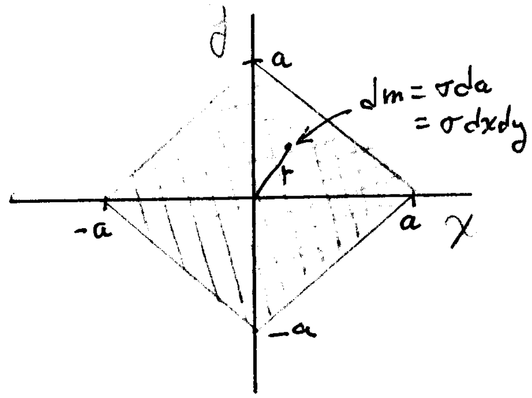
$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad (26)$$

I_{xx}, I_{yy}, & I_{zz} are the moments of inertia around the x, y, & z axes respectively. The off diagonal elements are called "products of inertia." (21) would be greatly simplified, if I were diagonal

Then
$$\begin{pmatrix} L_x' \\ L_y' \\ L_z' \end{pmatrix} = \begin{pmatrix} I_{x'} & 0 & 0 \\ 0 & I_{y'} & 0 \\ 0 & 0 & I_{z'} \end{pmatrix} \begin{pmatrix} \omega_x' \\ \omega_y' \\ \omega_z' \end{pmatrix} \qquad (27)$$

Then the diagonal elements are called the "principle moments of inertia." They are the eigenvalues of I.

As an example, consider the planar object below



The integral in (22) becomes a surface integral and $\rho = \sigma$ where $\sigma = M/4a^2$

then

$$I_{kj} = \frac{M}{4a^2} \int \left(r^2 \delta_{kj} - x_k x_j \right) d\sigma$$

with $r^2 = x^2 + y^2$

For example, with $j = k = 1$

$$I_{xx} = \frac{4M}{4a^2}\int_0^a\int_0^a (r^2 - x^2)\,dx\,dy = \frac{M}{a^2}\int_0^a\int_0^a y^2\,dx\,dy \tag{29}$$

$$I_{xx} = \frac{M}{a^2}(a)\frac{a^3}{3} = \frac{1}{3}Ma^2 \tag{30}$$

and
$$I_{xy} = \frac{4M}{4a^2}\int_{x=0}^{a}\int_{y=0}^{a}(-xy)\,dx\,dy = -\frac{1}{4}Ma^2 \tag{31}$$

and $\quad I_{xz} = 0 \quad$ since $z = 0$ (planar objects)

Similarly
$$I_{yy} = \frac{M}{a^2}\int_{x=0}^{a}\int_{y=0}^{a}(r^2 - y^2)\,dx\,dy = \frac{M}{a^2}\iint x^2\,dx\,dy$$

$$I_{yy} = \frac{1}{3}Ma^2$$

and
$$I_{yx} = \frac{M}{a^2}\int_0^a\int_0^a(-xy)\,dx\,dy = -\frac{1}{4}Ma^2$$

$$I_{yz} = 0$$

$$I_{zz} = \frac{M}{a^2}\int_0^a\int_0^a (x^2 + y^2 - \overset{=0}{z^2})\,dx\,dy = \frac{M}{a^2}\int_0^a\int_0^a x^2\,dx\,dy + y^2\,dx\,dy$$

$$I_{zz} = \frac{M}{a^2}\left[\left(\frac{1}{3}a^4 + \frac{1}{3}a^4\right)\right] = \frac{2}{3}Ma^2$$

$$I_{zx} = I_{zy} = 0$$

then
$$I = Ma^2\begin{pmatrix} \frac{1}{3} & -\frac{1}{4} & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix} = \frac{Ma^2}{12}\begin{pmatrix} 4 & -3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$